Introduction to Tensor Computation

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June 09-19, 2025

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- M-Product
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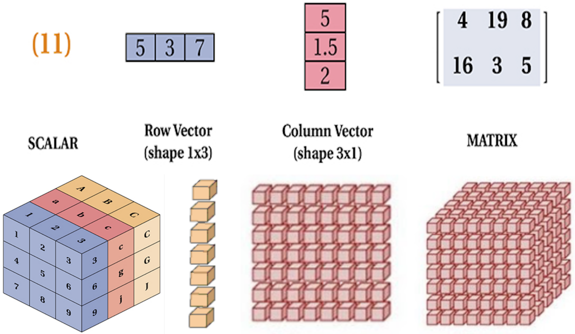
What is tensor?

- Tensor is a type of data structure used in ML to represent various kinds of objects including scalars, vectors, arrays, matrices and other tensors.
- Higher-dimensional generalization of a matrix (multidimensional arrays.)
- The order of a tensor is defined by the number of directions required to describe it.



Why Tensor

- Mathematical modeling of problems $\rightarrow$ solving multilinear systems.
- Standard matrix representation of data analysis is not enough to represent all the information content of the higher dimensional fields.
- Tensor (multiway arrays) is a natural representation for massive multidimensional data.

Examples

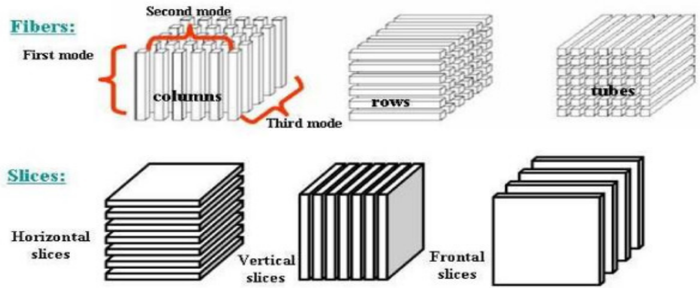
- In a stadium: indexed by section, row, seat
- Finance data: indexed by institute, year, month, day
- Geographical data: longitude, latitude, temperature, pressure, rainfall, time.
- Data on health status: blood analysis, immune system status, genetic background, nutrition, alcohol-tobacco-drug-consuption, operations.

Tensor Representations

- The higher order analogue of matrix rows and columns are called fibers.
- A matrix column is a mode-1 fiber and a matrix row is a mode-2 fiber.

For a third order tensor: x_{ijk}

- We have column $(x_{:jk})$, row $(x_{i:k})$, and tube $(x_{ij:})$ fibers.
- We have horizontal $(x_{i::})$, vertical $(x_{:j:})$ and frontal $(x_{::k})$ slices



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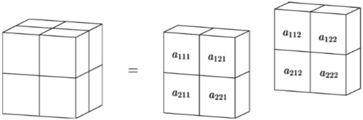
Example

☞ Consider a 4th order tensor $\mathscr{A} \in \mathbb{R}^{2\times2\times2\times3}$. Now fixing the 4th index, we can get three $2\times2\times2$ tensors. The elements of $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ are

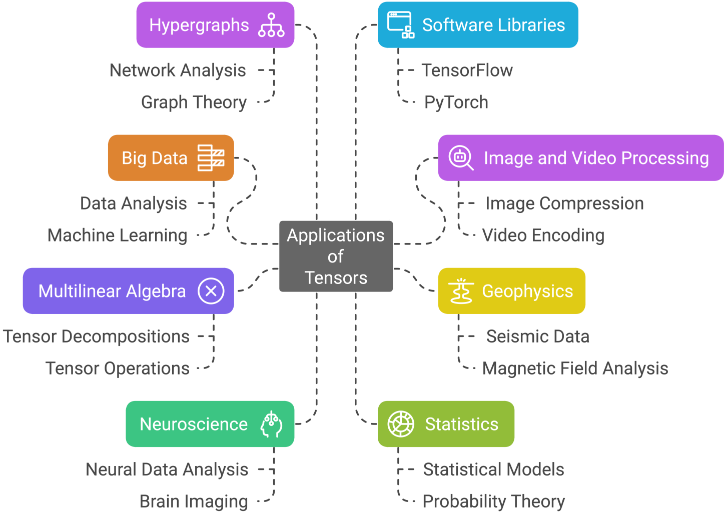
$$\begin{array}{lcllllllll} \mathscr{A}_1 & : & a_{1111} & a_{1211} & a_{2111} & a_{2211} & a_{1121} & a_{1221} & a_{2121} & a_{2221}, \\ \mathscr{A}_2 & : & a_{1112} & a_{1212} & a_{2112} & a_{2212} & a_{1122} & a_{1222} & a_{2122} & a_{2222}, \\ \mathscr{A}_3 & : & a_{1113} & a_{1213} & a_{2113} & a_{2213} & a_{1123} & a_{1223} & a_{2123} & a_{2223}, \end{array}$$

☞ A third-order tensor $\mathscr{A} \in \mathbb{R}^{2\times2\times2}$ looks like a cube of data, but the orientation of third-order tensors is not unique, i.e,

$\boxed{(\mathscr{A})_{i,j,1} \text{ and } (\mathscr{A})_{i,j,2}}$ or $\boxed{(\mathscr{A})_{1,i,j} \text{ and } (\mathscr{A})_{2,i,j}}$ or $\boxed{(\mathscr{A})_{i,1,j} \text{ and } (\mathscr{A})_{i,2,j}}$



Applications of Tensors in Various Domains



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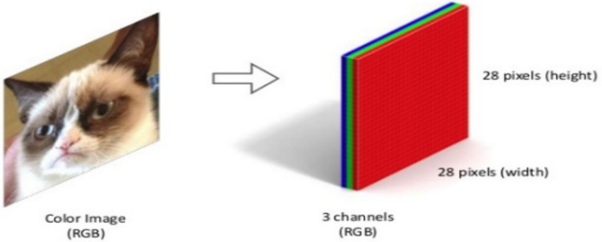
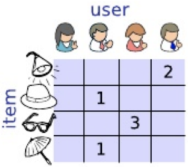
Example (Color Images)

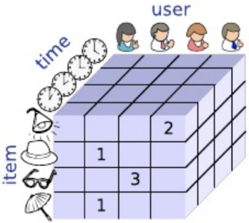


Example (Color Videos)

Example (Relational Data)

Relational data is a collection of relations among multiple objects





Relationship of pair $\Longleftrightarrow$ Matrix (2 dim array)

Relationships of 3-tuple $\Longleftrightarrow$ Tensor (3 dim array)

Tensor

Tensor representation is generally high-dimensional and large-scale.

EX: 1000 users $\times$ 1000 items $\times$ 1000 times = total 10^9 relationship

AIM $\longrightarrow$ To develop tensor based model for the applications

- Efficiency, scalability, productivity, and reliability

Tensor Computations

- Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, then $C = AB \in \mathbb{R}^{m\times p}$

$$c_{ij} = \sum_{k=1}^{k=n} a_{ik}b_{kj}, \text{ for } 1 \le i \le m, 1 \le j \le n,$$

- Suppose $\mathscr{A} = (a_{i_1 \dots i_p}) \in \mathbb{R}^{n_1\times\cdots\times n_p}, p > 2$ be the tensor of order p
- Here, if p is even (odd), then $\mathscr{A}$ is an even (odd) order tensor.
- Let $\boxed{\mathscr{A} = (a_{ijk}) \in \mathbb{R}^{3\times 2\times 3}, \ \mathscr{B} = (b_{ijk}) \in \mathbb{R}^{2\times 3\times 3}}$

Theory/Computation/Application of Tensors

- Tensor Multiplication
- Tensor Decomposition
- Tensor Inverse
- Image Deblurring and Compression
- Face Recognition
- High-Dimensional PDEs

Progress of Tensor Products

- n-Mode Product (2000): By L. Lathauwer et. al.
- T-Product (2011): By Kilmer and Martin
- Shao-Product (2013): By J. Shao et. al.
- C-Product (2015) & M-product (2015): By Kernfeld et. al.

☞ L. Lathauwer, B. Moor, J. Vandewalle A multilinear singular value decomposition. SIAM J. Matrix Anal. Appl.21(2000), no.4, 1253-1278.

☞ Shao, Jia-Yu. A general product of tensors with applications. Linear Algebra Appl. 2013;439(8):2350-2366.

☞ Kilmer, Misha E., and Carla D. Martin. Factorization strategies for third-order tensors. Linear Algebra Appl. 2011;435(3):641-658.

☞ E. Kernfeld, M. Kilmer, S. Aeron. Tensor-tensor products with invertible linear transforms. Linear Algebra Appl. 2015; 485:545-570.

☞ T. G. Kolda and B. W. Bader. Tensor Decompositions and Applications. SIAM Rev., 51(3):455-500, 2009.

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☞ E. Kernfeld, M. Kilmer, S. Aeron. Tensor-tensor products with invertible linear transforms. Linear Algebra Appl. 2015; 485:545-570.

☞ T. G. Kolda and B. W. Bader. Tensor Decompositions and Applications. SIAM Rev., 51(3):455-500, 2009.

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Tensor Multiplication

Matrix multiplication

If A is an $M \times N$ matrix and B is an $N \times P$ matrix, then

$$AB = \sum_{k=1}^{N} a_{ik} b_{kj} \quad 1 \leq i \leq M, \;\; 1 \leq j \leq P, \;\; 1 \leq k \leq N \tag{1}$$

Tensor multiplication (Einstein product)

For $\mathscr{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$ and $\mathscr{B} \in \mathbb{C}^{K_1 \times \cdots \times K_N \times J_1 \times \cdots \times J_M}$, the Einstein product of tensors $\mathscr{A}$ and $\mathscr{B}$ is defined by the operation $*_N$ via

$$(\mathscr{A} *_N \mathscr{B})_{i_1 \ldots i_N j_1 \ldots j_M} = \sum_{k_1 \ldots k_N} a_{i_1 \ldots i_N k_1 \ldots k_N} b_{k_1 \ldots k_N j_1 \ldots j_M}. \tag{2}$$

Example: $\mathscr{A} \in \mathbb{R}^{3 \times 4 \times 5 \times 6}$. $\mathscr{B} \in \mathbb{R}^{5 \times 6 \times 7 \times 8}$, then

$$\mathscr{A} *_2 \mathscr{B} = \sum_{j=1}^{5} \sum_{k=1}^{6} a_{ijkl} b_{klmn}, \;\; \text{where } \mathscr{A} *_2 \mathscr{B} \in \mathbb{R}^{3 \times 4 \times 7 \times 8} \tag{3}$$

☞ ☞ A. Einstein The foundation of the general theory of relativity, *Annals of Physics* (1916).

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Definitions

Definition (Transpose)

The transpose of $\mathscr{A} \in \mathbb{R}^{I\times J\times I\times J}$ is a tensor $\mathscr{B}$ which has entries $b_{ijkl} = a_{klij}$. We denote the transpose of $\mathscr{A}$ as $\mathscr{B} = \mathscr{A}^T$.
If $\mathscr{A} \in \mathbb{R}^{I_1\times I_2\ldots\times I_N\times J_1\times\ldots\times J_N}$, then
$(\mathscr{B})_{i_1,i_2,\ldots,i_n j_1 j_2,\ldots,j_n} = (\mathscr{A})^T_{j_1 j_2,\ldots,j_n,i_1,i_2,\ldots,i_n}$ is the transpose of $\mathscr{A}$.

Definition (Symmetric tensor)

A tensor $\mathscr{A} \in \mathbb{R}^{I_1\times I_2\ldots\times I_N\times J_1\times\ldots\times J_N}$ is symmetric if $\mathscr{A} = \mathscr{A}^T$, that is,
$a_{i_1,i_2,\ldots,i_n j_1 j_2,\ldots,j_n} = a_{j_1 j_2,\ldots,j_n,i_1,i_2,\ldots,i_n}$.

☞ M. Brazell, N. Li, C. Navasca, et al. Solving multilinear systems via tensor inversion. SIAM J. Matrix Anal Appl. 2013;34(2):542-570.

Example

➢ Let $\mathscr{A} \in \mathbb{R}^{2\times2\times2\times2}$, with entries

$$\mathscr{A}_{:,:,1,1} = \begin{bmatrix} 1 & 0 \\ 5 & -1 \end{bmatrix} \mathscr{A}_{:,:,1,2} = \begin{bmatrix} 3 & 9 \\ 7 & -3 \end{bmatrix} \mathscr{A}_{:,:,2,1} = \begin{bmatrix} 2 & 5 \\ 6 & -2 \end{bmatrix} \mathscr{A}_{:,:,2,2} = \begin{bmatrix} 4 & 3 \\ 8 & -4 \end{bmatrix}$$

Then $\mathscr{B} = \mathscr{A}^T$ with entries

$$\mathscr{B}_{:,:,1,1} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \mathscr{B}_{:,:,1,2} = \begin{bmatrix} 0 & 9 \\ 5 & 3 \end{bmatrix} \mathscr{B}_{:,:,2,1} = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} \mathscr{B}_{:,:,2,2} = \begin{bmatrix} -1 & -3 \\ -2 & -4 \end{bmatrix}$$

Definition

Definition (Identity tensor)

The identity tensor $\mathscr{I}$ is $\mathscr{I}_{i_1 i_2 j_1 j_2} = \delta_{i_1 j_1} \delta_{i_2 j_2}$, where $\delta_{lk} = \begin{cases} 1, & l = k \\ 0, & l \neq k. \end{cases}$

It generalizes to an $2N$th order identity tensor,

$$(\mathscr{I})_{i_1 i_2 \ldots i_N j_1 j_2 \ldots j_N} = \prod_{k=1}^{N} \delta_{i_k j_k}. \tag{4}$$

➢ Let $\mathscr{I} \in \mathbb{R}^{2\times 2\times 2\times 2}$, then

$$\mathscr{I}_{:,:,1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathscr{I}_{:,:,1,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathscr{I}_{:,:,2,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathscr{I}_{:,:,2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition

Definition (Diagonal tensor)

A tensor $\mathscr{D} \in \mathbb{R}^{I\times J\times I\times J}$ is called diagonal if $d_{ijkl} = 0$ when $i \neq k$ and $j \neq l$.

- Let $\mathscr{I} \in \mathbb{R}^{2\times 2\times 2\times 2}$, then

$$\mathscr{D}_{:,:,1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathscr{D}_{:,:,1,2} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \mathscr{D}_{:,:,2,1} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \mathscr{D}_{:,:,2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Tensor computations

A Common Framework for Tensor Computations

- Turn tensor $\mathscr{A}$ into a matrix A.
- Through matrix computations, discover things about A.
- Draw conclusions about tensor $\mathscr{A}$ based on what is learned about matrix A.

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Tensor computations

Definition (Transformation)

Define the transformation $f : \mathbb{T}_{I_1,I_2,I_1,I_2}(\mathbb{R}) \longrightarrow \mathbb{M}_{I_1 I_2, I_1 I_2}(\mathbb{R})$ with $f(\mathscr{A}) = A$ is defined component-wise as

$$(\mathscr{A})_{i_1 i_2 i_1 i_2} \xrightarrow{f} (A)_{[i_1+(i_2-1)I_1][i_1+(i_2-1)I_1]} \tag{5}$$

where $\mathbb{T}_{I_1,I_2,I_1,I_2}(\mathbb{R}) = \{\mathscr{A} \in \mathscr{R}^{I_1\times I_2\times I_1\times I_2} \ : \ det(f(\mathscr{A})) \neq 0\}$ and $A \in \mathbb{M}_{I_1 I_2, I_1 I_2}(\mathbb{R})$.
In general, for $\mathscr{A} \in \mathbb{T}_{I_1,\ldots,I_N,J_1,\ldots,J_N}(\mathbb{R})$ and $A \in \mathbb{M}_{I_1\ldots I_N, J_1\ldots J_N}(\mathbb{R})$, then

$$(\mathscr{A})_{i_1 i_2 \ldots i_N j_1 j_2 \ldots j_N} \xrightarrow{f} (A)_{[i_1+\sum_{k=2}^{N}(i_k-1)\prod_{l=1}^{k-1} I_l][j_1+\sum_{k=2}^{N}(j_k-1)\prod_{l=1}^{k-1} J_l]} \tag{6}$$

Theorem-1

Let f be the map defined in (5). Then the following properties hold:

(1) The map f is a bijection. Moreover, there exists a bijective inverse map $f^{-1} : \mathbb{M}_{I_1 I_2, I_1 I_2}(\mathbb{R}) \rightarrow \mathbb{T}_{I_1,I_2,I_1,I_2}(\mathbb{R})$.

(2) The map satisfies $f(\mathscr{A} *_2 \mathscr{B}) = f(\mathscr{A}) \cdot f(\mathscr{B})$, where ‘ $\cdot$ ‘ refers to matrix multiplication

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Consider a tensor $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{2\times3\times2\times3}$

$$a_{ij11} = \begin{pmatrix} a_{1111} & a_{1211} & a_{1311} \\ a_{2111} & a_{2211} & a_{2311} \end{pmatrix}, a_{ij12} = \begin{pmatrix} a_{1112} & a_{1212} & a_{1312} \\ a_{2112} & a_{2212} & a_{2312} \end{pmatrix},$$

$$a_{ij13} = \begin{pmatrix} a_{1113} & a_{1213} & a_{1313} \\ a_{2113} & a_{2213} & a_{2313} \end{pmatrix}, a_{ij21} = \begin{pmatrix} a_{1121} & a_{1221} & a_{1321} \\ a_{2121} & a_{2221} & a_{2321} \end{pmatrix},$$

$$a_{ij22} = \begin{pmatrix} a_{1122} & a_{1222} & a_{1322} \\ a_{2122} & a_{2222} & a_{2322} \end{pmatrix}, a_{ij23} = \begin{pmatrix} a_{1123} & a_{1223} & a_{1323} \\ a_{2123} & a_{2223} & a_{2323} \end{pmatrix},$$

$$\mathscr{A} = \begin{bmatrix} a_{1111} & a_{1121} & a_{1112} & a_{1122} & a_{1113} & a_{1123} \\ a_{2111} & a_{2121} & a_{2112} & a_{2122} & a_{2113} & a_{2123} \\ a_{1211} & a_{1221} & a_{1212} & a_{1222} & a_{1213} & a_{1223} \\ a_{2211} & a_{2221} & a_{2212} & a_{2222} & a_{2213} & a_{2223} \\ a_{1311} & a_{1321} & a_{1312} & a_{1322} & a_{1313} & a_{1323} \\ a_{2311} & a_{2321} & a_{2312} & a_{2322} & a_{2313} & a_{2323} \end{bmatrix}$$

Drawback of the Einstein product

Definition (Transformation)

Define the transformation $f : \mathbb{T}_{I_1,I_2,I_1,I_2}(\mathbb{R}) \longrightarrow \mathbb{M}_{I_1I_2,I_1I_2}(\mathbb{R})$ with $f(\mathscr{A}) = \mathbf{A}$ is defined component-wise as

$$(\mathscr{A})_{i_1i_2i_1i_2} \xrightarrow{f} (\mathbf{A})_{[i_1+(i_2-1)I_1][i_1+(i_2-1)I_1]} \tag{7}$$

where $\mathscr{A} \in \mathbb{T}_{I_1,I_2,I_1,I_2}(\mathbb{R})$ and $\mathbf{A} \in \mathbb{M}_{I_1I_2,I_1I_2}(\mathbb{R})$.

- Using the above Definition the tensor $\mathscr{A} \in \mathbb{R}^{I_1\times I_2\times I_1\times I_2}$, which can be extended only to any square tensor, i.e., for $\mathscr{A} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$.
- But the extension of an arbitrary-order tensor is impossible, since f is not homomorphism for even-order and/or arbitrary-order tensors.
- The Einstein product is not defined for the following two even-order tensors, $\mathscr{A} \in \mathbb{R}^{I_1\times I_2\times J_1\times J_2}$ and $\mathscr{B} \in \mathbb{R}^{I_1\times I_2\times J_1\times J_2}$, i.e., $\mathscr{A} *_2 \mathscr{B}$ is not defined.

☞ R Behera, S Maji, and R. N. Mohapatra, Weighted Moore-Penrose inverses of arbitrary-order tensors, Computational and Applied Mathematics, (2020), 39:284

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Efficient Solution of the Poisson problem

➢ Consider d-dimensional Poisson equation

$$\begin{aligned} -\nabla^2 u &= f \text{ in } \Omega \subset \mathbb{R}^d \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

The discretized d-dimensional Poisson equation can be written as follows:

$$Ax = b, \text{ where the matrix } A \in \mathbb{R}^{n^d \times n^d}, \text{ vectors } x, b \in \mathbb{R}^{n^d \times 1}$$

➢ The traditional matrix- and vector-based approach is impractical to solve

Example: Tensor representation of the 3D Poisson problem

Consider $d = 3$ and interior grid points $n = 100$ with Dirichlet boundary condition, then we obtain matrix $A \in \mathbb{R}^{10^6 \times 10^6}$, vectors $x \in \mathbb{R}^{10^6 \times 1}$ and $b \in \mathbb{R}^{10^6 \times 1}$.

$\mathscr{A} * \mathscr{X} = \mathscr{B}$, $\mathscr{A} \in \mathbb{R}^{100\times100\times100\times100\times100\times100}$, $\mathscr{X} \in \mathbb{R}^{100\times100\times100}$, and $\mathscr{B} \in \mathbb{R}^{100\times100\times100}$,

Ref: R. Behera, A. Nandi, J. Sahoo. Further results on the Drazin inverse of even-order tensors. Numer Linear Algebra Appl. (2020) e2317. DOI: 10.1002/nla.2317

Poisson problem

Multilinear systems in Poisson problems

Consider the two-dimension Poisson problem on $\Omega = \{(x,y) : 0 < x,y < 1\}$ with boundary $\partial\Omega$. For a given function $f(x,y)$ we have

$$\begin{aligned} -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} &= f(x,y) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

Using 5-point stencil central difference scheme on a discretizing the unit square domain with n interior nodes, we obtain the following multilinear system

$$\mathscr{A} *_2 \mathscr{X} = \mathscr{B}, \quad \text{where } \mathscr{A} \in \mathbb{R}^{n\times n\times n\times n},\ \mathscr{X} \in \mathbb{R}^{n\times n}, \text{ and } \mathscr{B} \in \mathbb{R}^{n\times n},$$

$$\text{where the tensor } \mathscr{A} \text{ is of the form } \quad \mathscr{A} = \mathscr{I}_n \otimes \mathscr{P}_n + \mathscr{P}_n \otimes \mathscr{I}_n, \tag{8}$$

☞ Here $\mathscr{I}_n$ is the second order identity tensor and $\mathscr{P}_n$ is also a second order tensor of the form, $\mathscr{P}_n = tridiagonal(-1, 2, -1)$.

Poisson problem

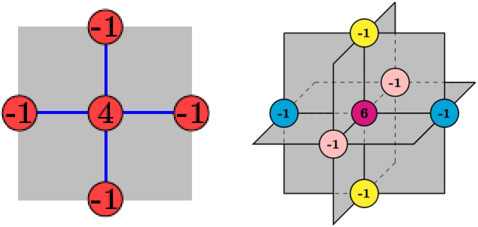

Figure: 5-point Stencil and 7-point Stencil

By applying 7-point stencil formula for 3-dimensional Poisson equation with same boundary conditions, we obtain the following tensor equation

$$\mathscr{A} *_3 \mathscr{X} = \mathscr{B},\ \mathscr{A} \in \mathbb{R}^{n\times n\times n\times n\times n\times n},\ \mathscr{X} \in \mathbb{R}^{n\times n\times n},\ \text{and } \mathscr{B} \in \mathbb{R}^{n\times n\times n},$$

$$\text{where } \mathscr{A} = \mathscr{P}_n \otimes \mathscr{I}_n \otimes \mathscr{I}_n + \mathscr{I}_n \otimes \mathscr{P}_n \otimes \mathscr{I}_n + \mathscr{I}_n \otimes \mathscr{I}_n \otimes \mathscr{P}_n. \tag{9}$$

Poisson problem

Extending, the same idea to 4-dimensional Poisson problem, we obtain the following multilinear system

$$\mathscr{A} *_4 \mathscr{X} = \mathscr{B}, \ \mathscr{A} \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{8 \ \text{times}}}, \ \mathscr{X} \in \mathbb{R}^{n \times n \times n \times n}, \text{ and } \mathscr{B} \in \mathbb{R}^{n \times n \times n \times n},$$

where the tensor $\mathscr{A}$ is the following form

$$\mathscr{A} = \mathscr{P}_n \otimes \mathscr{I}_n \otimes \mathscr{I}_n \otimes \mathscr{I}_n + \mathscr{I}_n \otimes \mathscr{P}_n \otimes \mathscr{I}_n \otimes \mathscr{I}_n + \mathscr{I}_n \otimes \mathscr{I}_n \otimes \mathscr{P}_n \otimes \mathscr{I}_n + \mathscr{I}_n \otimes \mathscr{I}_n + \mathscr{I}_n \otimes \mathscr{P}_n.$$

In the light of the above Eqs (8), (9) and (10) one can generate the tensor $\mathscr{A}$ to solve higher dimensional Poisson problem.

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Unfolding of a 3rd order Tensor

➢ Let $\mathscr{A} \in \mathbb{R}^{3\times4\times2}$, with entries (frontal slices)

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \mathscr{A}(:,:,2) = \begin{bmatrix} 13 & 16 & 19 & 22 \\ 14 & 17 & 20 & 23 \\ 15 & 18 & 21 & 24 \end{bmatrix}.$$

- Mode-1 unfolding of $\mathscr{A}$ is a matrix of order 3×8 and entries are given by

$$\begin{bmatrix} 1 & 4 & 7 & 10 & 13 & 16 & 19 & 22 \\ 2 & 5 & 8 & 11 & 14 & 17 & 20 & 23 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 \end{bmatrix}$$

- Mode-2 unfolding of $\mathscr{A}$ is a matrix of order 2×6 and entries are given by

$$\begin{bmatrix} 1 & 2 & 3 & 13 & 14 & 15 \\ 4 & 5 & 6 & 16 & 17 & 18 \\ 7 & 8 & 9 & 19 & 20 & 21 \\ 10 & 11 & 12 & 22 & 23 & 24 \end{bmatrix}$$

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Unfolding of a 3rd order Tensor

- Let $\mathscr{A} \in \mathbb{R}^{3\times4\times2}$, with entries (frontal slices)

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \mathscr{A}(:,:,2) = \begin{bmatrix} 13 & 16 & 19 & 22 \\ 14 & 17 & 20 & 23 \\ 15 & 18 & 21 & 24 \end{bmatrix}.$$

- Mode-3 unfolding of $\mathscr{A}$ is a matrix of order 2×12 and entries are given by

$$A_{(3)} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \end{vmatrix}$$

Unfolding of a 3rd order Tensor

➢ Let $\mathscr{A} \in \mathbb{R}^{3\times4\times2}$, with entries (frontal slices)

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \mathscr{A}(:,:,2) = \begin{bmatrix} 13 & 16 & 19 & 22 \\ 14 & 17 & 20 & 23 \\ 15 & 18 & 21 & 24 \end{bmatrix}.$$

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3-Mode Product

Definition

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ be a tensor and $M \in \mathbb{R}^{p\times p}$ be a matrix. The 3-mode product of $\mathscr{A}$ with M is denoted by $\mathscr{A} \times_3 M \in \mathbb{R}^{m\times n\times p}$ and element-wise defined as

$$(\mathscr{A} \times_3 B)_{ijk} = \sum_{s=1}^{p} a_{ijs} b_{ks} \quad i = 1,2,\dots,m,\ j = 1,2,\dots,n,\ k = 1,2\dots,p.$$

Note: From here onwards we will assume M to be invertible.

☞ Kolda, Tamara G., and Brett W. Bader. Tensor decompositions and applications. SIAM review. 2009; 51(3):455-500.

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Note: From here onwards we will assume M to be invertible.

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3-Mode Product

Example

Let $\mathscr{A} \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \ \mathscr{A}(:,:,2) = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}, \ \text{and } M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Evaluate: $MA_{(3)} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 10 & 12 & 14 & 16 \end{bmatrix}$

Now $\mathscr{B} = \mathscr{A} \times_3 M \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\mathscr{B}(:,:,1) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \ \mathscr{B}(:,:,2) = \begin{bmatrix} 10 & 14 \\ 12 & 16 \end{bmatrix}.$$

3-Mode Product

Example

Let $\mathscr{A} \in \mathbb{R}^{2\times 2\times 2}$ with entries

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$$\mathscr{B}(:,:,1) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \ \mathscr{B}(:,:,2) = \begin{bmatrix} 10 & 14 \\ 12 & 16 \end{bmatrix}.$$

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Example

Let $\mathscr{A} \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \mathscr{A}(:,:,2) = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}, \text{ and } M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Evaluate: $MA_{(3)} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 10 & 12 & 14 & 16 \end{bmatrix}$

Now $\mathscr{B} = \mathscr{A} \times_3 M \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\mathscr{B}(:,:,1) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \mathscr{B}(:,:,2) = \begin{bmatrix} 10 & 14 \\ 12 & 16 \end{bmatrix}.$$

Face-wise Product

Tensor-Tensor Multiplication:

Definition

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ and $B \in \mathbb{R}^{n\times k\times p}$ be two tensors. The face-wise product of $\mathscr{A}$ and $\mathscr{B}$ is denoted by $\mathscr{A} \triangle \mathscr{B} \in \mathbb{R}^{m\times k\times p}$ and element-wise defined as

$$(\mathscr{A} \triangle \mathscr{B})(:,:,i) = \mathscr{A}(:,:,i)\mathscr{B}(:,:,i), \; i = 1,2\ldots,p.$$

☞ E. Kernfeld, M. Kilmer, S. Aeron. Tensor-tensor products with invertible linear transforms. Linear Algebra Appl. 2015; 485:545-570.

Face-wise Product

Example Let $\mathscr{A} \in \mathbb{R}^{2\times3\times2}$ and $\mathscr{B} \in \mathbb{R}^{3\times2\times2}$ with respective frontal slices are

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \mathscr{B}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathscr{A}(:,:,2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \mathscr{B}(:,:,2) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Thus we have $(\mathscr{A} \triangle \mathscr{B})(:,:,1) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $(\mathscr{A} \triangle \mathscr{B})(:,:,2) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

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Face-wise Product

Example Let $\mathscr{A} \in \mathbb{R}^{2\times 3\times 2}$ and $\mathscr{B} \in \mathbb{R}^{3\times 2\times 2}$ with respective frontal slices are

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \mathscr{B}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathscr{A}(:,:,2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \mathscr{B}(:,:,2) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Thus we have $(\mathscr{A} \triangle \mathscr{B})(:,:,1) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $(\mathscr{A} \triangle \mathscr{B})(:,:,2) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

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M-Product

Tensor-Tensor Multiplication:

Definition

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ and $B \in \mathbb{R}^{n\times k\times p}$ be two tensors and $M \in \mathbb{R}^{p\times p}$. The M-product of $\mathscr{A}$ and $\mathscr{B}$ is denoted by $\mathscr{A} *_M \mathscr{B} \in \mathbb{R}^{m\times k\times p}$ and defined as

$$\mathscr{A} *_M \mathscr{B} = [(\mathscr{A} \times_3 M)\triangle(\mathscr{B} \times_3 M)] \times_3 M^{-1}.$$

☞ E. Kernfeld, M. Kilmer, S. Aeron. Tensor-tensor products with invertible linear transforms. Linear Algebra Appl. 2015; 485:545-570.

Algorithm for M-Product

Algorithm 1: M-Product of $\mathscr{A}$ and $\mathscr{B}$

Input $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$, $\mathscr{A} \in \mathbb{R}^{n\times k\times p}$ and $M \in \mathbb{R}^{p\times p}$.

Compute $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$, $\tilde{\mathscr{B}} = \mathscr{B} \times_3 M$

for $i \leftarrow 1$ to p **do**

$\tilde{\mathscr{C}}(:,:,i) = \tilde{\mathscr{A}}(:,:,i)\tilde{\mathscr{B}}(:,:,i)$

end for

Compute $\mathscr{C} = \tilde{\mathscr{C}} \times_3 M^{-1}$

return $\mathscr{C} = \mathscr{A} *_M \mathscr{B}$

Note: From here onwards, we will use the tilde notation with respect to M (for example $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$).

Algorithm for M-Product

Algorithm 2: M-Product of $\mathscr{A}$ and $\mathscr{B}$

Input $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$, $\mathscr{A} \in \mathbb{R}^{n\times k\times p}$ and $M \in \mathbb{R}^{p\times p}$.
Compute $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$, $\tilde{\mathscr{B}} = \mathscr{B} \times_3 M$
for $i \leftarrow 1$ to p **do**
    $\tilde{\mathscr{C}}(:,:,i) = \tilde{\mathscr{A}}(:,:,i)\tilde{\mathscr{B}}(:,:,i)$
end for
Compute $\mathscr{C} = \tilde{\mathscr{C}} \times_3 M^{-1}$
return $\mathscr{C} = \mathscr{A} *_M \mathscr{B}$

Note: From here onwards, we will use the tilde notation with respect to M (for example $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$).

Algorithm for M-Product

Algorithm 3: M-Product of $\mathscr{A}$ and $\mathscr{B}$

Input $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$, $\mathscr{A} \in \mathbb{R}^{n\times k\times p}$ and $M \in \mathbb{R}^{p\times p}$.
Compute $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$, $\tilde{\mathscr{B}} = \mathscr{B} \times_3 M$
for $i \leftarrow 1$ to p **do**
  $\tilde{\mathscr{C}}(:,:,i) = \tilde{\mathscr{A}}(:,:,i)\tilde{\mathscr{B}}(:,:,i)$
end for
Compute $\mathscr{C} = \tilde{\mathscr{C}} \times_3 M^{-1}$
return $\mathscr{C} = \mathscr{A} *_M \mathscr{B}$

Note: From here onwards, we will use the tilde notation with respect to M (for example $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$).

Algorithm for M-Product

Algorithm 4: M-Product of $\mathscr{A}$ and $\mathscr{B}$

1 **Input** $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$, $\mathscr{A} \in \mathbb{R}^{n\times k\times p}$ and $M \in \mathbb{R}^{p\times p}$.
2 **Compute** $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$, $\tilde{\mathscr{B}} = \mathscr{B} \times_3 M$
3 **for** $i \leftarrow 1$ to p **do**
4 $\tilde{\mathscr{C}}(:,:,i) = \tilde{\mathscr{A}}(:,:,i)\tilde{\mathscr{B}}(:,:,i)$
5 **end for**
6 **Compute** $\mathscr{C} = \tilde{\mathscr{C}} \times_3 M^{-1}$
7 **return** $\mathscr{C} = \mathscr{A} *_M \mathscr{B}$

Note: From here onwards, we will use the tilde notation with respect to M (for example $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$).

M-Product(Example)

Let $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathscr{A}, \mathscr{B} \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathscr{A}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \mathscr{B}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathscr{B}(:,:,2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

➢ Find $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M \in \mathbb{R}^{2\times 2\times 2}$ and $\tilde{\mathscr{B}} = \mathscr{B} \times_3 M \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\tilde{\mathscr{A}}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tilde{\mathscr{A}}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \tilde{\mathscr{B}}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tilde{\mathscr{B}}(:,:,2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$(\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}})(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, (\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}})(:,:,2) = \begin{bmatrix} 0 & 0 \\ 4 & 4 \end{bmatrix}.$$

Thus $\mathscr{A} *_M \mathscr{B} = (\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}}) \times_3 M^{-1}$ is given by

$$\mathscr{A} *_M \mathscr{B}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathscr{A} *_M \mathscr{B}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$

M-Product(Example)

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- Find $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M \in \mathbb{R}^{2\times 2\times 2}$ and $\tilde{\mathscr{B}} = \mathscr{B} \times_3 M \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\tilde{\mathscr{A}}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tilde{\mathscr{A}}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \tilde{\mathscr{B}}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tilde{\mathscr{B}}(:,:,2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$(\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}})(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, (\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}})(:,:,2) = \begin{bmatrix} 0 & 0 \\ 4 & 4 \end{bmatrix}.$$

Thus $\mathscr{A} *_M \mathscr{B} = (\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}}) \times_3 M^{-1}$ is given by

$$\mathscr{A} *_M \mathscr{B}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathscr{A} *_M \mathscr{B}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$

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M-Product(Example)

Let $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathscr{A}$, $\mathscr{B} \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathscr{A}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \mathscr{B}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathscr{B}(:,:,2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

➢ Find $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M \in \mathbb{R}^{2\times 2\times 2}$ and $\tilde{\mathscr{B}} = \mathscr{B} \times_3 M \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\tilde{\mathscr{A}}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tilde{\mathscr{A}}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \tilde{\mathscr{B}}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tilde{\mathscr{B}}(:,:,2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$(\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}})(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, (\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}})(:,:,2) = \begin{bmatrix} 0 & 0 \\ 4 & 4 \end{bmatrix}.$$

Thus $\mathscr{A} *_M \mathscr{B} = (\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}}) \times_3 M^{-1}$ is given by

$$\mathscr{A} *_M \mathscr{B}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathscr{A} *_M \mathscr{B}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$

M-Product(Example)

Let $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathscr{A}$, $\mathscr{B} \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathscr{A}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \mathscr{B}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathscr{B}(:,:,2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

➢ Find $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M \in \mathbb{R}^{2\times 2\times 2}$ and $\tilde{\mathscr{B}} = \mathscr{B} \times_3 M \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\tilde{\mathscr{A}}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tilde{\mathscr{A}}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \tilde{\mathscr{B}}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tilde{\mathscr{B}}(:,:,2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$(\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}})(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, (\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}})(:,:,2) = \begin{bmatrix} 0 & 0 \\ 4 & 4 \end{bmatrix}.$$

Thus $\mathscr{A} *_M \mathscr{B} = (\tilde{\mathscr{A}} \triangle \tilde{\mathscr{B}}) \times_3 M^{-1}$ is given by

$$\mathscr{A} *_M \mathscr{B}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathscr{A} *_M \mathscr{B}(:,:,2) = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$

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Definition

Definition (Identity tensor)

Let $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ and $M \in \mathbb{R}^{p\times p}$. Then $\mathscr{A}$ is called an identity tensor if $\tilde{\mathscr{A}}(:,:,i)$ is an identity matrix of order $m \times m$, for each $i = 1, 2, \ldots, p$

- Let $\mathscr{A} \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathscr{A}(:,:,2) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now see the frontal slices of $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$ that is

$$\tilde{\mathscr{A}}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \tilde{\mathscr{A}}(:,:,2).$$

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Definition (Identity tensor)

Let $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ and $M \in \mathbb{R}^{p\times p}$. Then $\mathscr{A}$ is called an identity tensor if $\tilde{\mathscr{A}}(:,:,i)$ is an identity matrix of order $m \times m$, for each $i = 1, 2, \ldots, p$

- Let $\mathscr{A} \in \mathbb{R}^{2\times 2\times 2}$ with entries

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Definition

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➢ Let $\mathscr{A} \in \mathbb{R}^{2\times 2\times 2}$ with entries

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathscr{A}(:,:,2) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now see the frontal slices of $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$ that is

$$\tilde{\mathscr{A}}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \tilde{\mathscr{A}}(:,:,2).$$

Definition

Definition (Diagonal tensor)

Let $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ and $M \in \mathbb{R}^{p\times p}$. Then $\mathscr{A}$ is called diagonal if $\tilde{\mathscr{A}}(:,:,i)$ is diagonal, for each $i = 1, 2, \ldots, p$

Definition (Conjugate Transpose)

Let $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ and $M \in \mathbb{R}^{p\times p}$. The elements of $\mathscr{A}^*$ is obtained as follows: **Compute** $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$

for $i \leftarrow 1$ to p **do**

$\tilde{\mathscr{C}}(:,:,i) = (\tilde{\mathscr{A}}(:,:,i))^*$

end for

Compute $\mathscr{A}^* = \tilde{\mathscr{C}} \times_3 M^{-1}$

Definition

Definition (Diagonal tensor)

Let $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ and $M \in \mathbb{R}^{p\times p}$. Then $\mathscr{A}$ is called diagonal if $\tilde{\mathscr{A}}(:,:,i)$ is diagonal, for each $i = 1, 2, \ldots, p$

Definition (Conjugate Transpose)

Let $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ and $M \in \mathbb{R}^{p\times p}$. The elements of $\mathscr{A}^*$ is obtained as follows: **Compute** $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$

for $i \leftarrow 1$ to p **do**

$\tilde{\mathscr{C}}(:,:,i) = (\tilde{\mathscr{A}}(:,:,i))^*$

end for

Compute $\mathscr{A}^* = \tilde{\mathscr{C}} \times_3 M^{-1}$

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Definition

Let $\mathbf{c} \in \mathbb{R}^n$. If $\mathbf{c} = [c_1, c_2, \cdots, c_n]^T$, then the circulant matrix is:

$$Circ(\mathbf{c}) = \begin{bmatrix} c_1 & c_n & \cdots & c_2 \\ c_2 & c_1 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \vdots \\ c_n & c_{n-1} & \cdots & c_1 \end{bmatrix}$$

Similarly, if $C_1, \ldots, C_n$ are $n_1 \times n_2$ real matrices, then

$$Circ(C_1, \ldots, C_n) = \begin{bmatrix} C_1 & C_n & \cdots & C_2 \\ C_2 & C_1 & \cdots & C_3 \\ \vdots & \vdots & \vdots & \vdots \\ C_n & C_{n-1} & \cdots & C_1 \end{bmatrix}.$$

- Let $\mathscr{A} = (a_{i_1 \ldots i_p}) \in \mathbb{R}^{n_1 \times \cdots \times n_p}, p > 1$ be the tensor of order p
- For $(i = 1, \ldots, n_p)$, let $\mathscr{A}_i \in \mathbb{R}^{n_1 \times \cdots \times n_{p-1}}$ be the tensor of order $(p-1)$.
- Here $\mathscr{A}_i$ is created by holding the pth index of $\mathscr{A}$ fixed at i.

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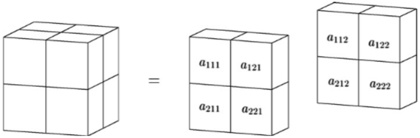
Examples

- Consider a 4th order tensor $\mathscr{A} \in \mathbb{R}^{2\times2\times2\times3}$. Now fixing the 4th index of $\mathscr{A}$.
- We can get three $2\times2\times2$ tensors. The elements of $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ are

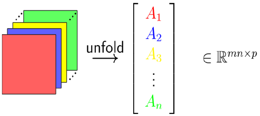
$$\begin{array}{lcllllllll}
\mathscr{A}_1 & : & a_{1111} & a_{1211} & a_{2111} & a_{2211} & a_{1121} & a_{1221} & a_{2121} & a_{2221}, \\
\mathscr{A}_2 & : & a_{1112} & a_{1212} & a_{2112} & a_{2212} & a_{1122} & a_{1222} & a_{2122} & a_{2222}, \\
\mathscr{A}_3 & : & a_{1113} & a_{1213} & a_{2113} & a_{2213} & a_{1123} & a_{1223} & a_{2123} & a_{2223},
\end{array}$$

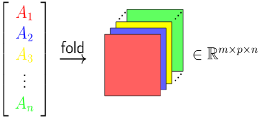
- A third-order tensor $\mathscr{A} \in \mathbb{R}^{2\times2\times2}$ looks like a cube of data.
- But the orientation of third-order tensors is not unique.
- i.e, $\boxed{(\mathscr{A})_{i,j,1} \text{ and } (\mathscr{A})_{i,j,2}}$ or $\boxed{(\mathscr{A})_{1,i,j} \text{ and } (\mathscr{A})_{2,i,j}}$ or $\boxed{(\mathscr{A})_{i,1,j} \text{ and } (\mathscr{A})_{i,2,j}}$.



t-Product

- The unfold transforms an $m \times p \times n$ tensor in to $mn \times p$ matrix
- fold(.) is the inverse operation, which takes an $mn \times p$ matrix and returns an $m \times p \times n$ tensor. Thus, $fold(unfold(\mathscr{A})) = \mathscr{A}$





Product of two tensors $\mathscr{A} \in \mathbb{R}^{m \times p \times n}$ and $\mathscr{B} \in \mathbb{R}^{p \times q \times n}$ is defined as

$$\mathscr{A} * \mathscr{B} = \text{fold}(\text{Circ}(unfold(\mathscr{A})) * unfold(\mathscr{B})), \quad \text{where } \mathscr{A} * \mathscr{B} \in \mathbb{R}^{m \times q \times n}$$

☞ **Example:** Consider two tensors $\mathscr{A} \in \mathbb{R}^{m \times p \times 3}$ and $\mathscr{B} \in \mathbb{R}^{p \times q \times 3}$ then

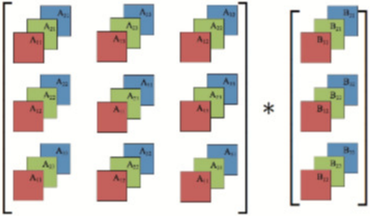
$$\mathscr{A} * \mathscr{B} = \text{fold}\left(\begin{bmatrix} A_1 & A_3 & A_2 \\ A_2 & A_1 & A_3 \\ A_3 & A_2 & A_1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}\right) \in \mathbb{R}^{m \times q \times 3}$$

☞ C. Martin, R. Shafer, and B. Larue. SIAM J. Sci. Comput. (2013) 35(1) 474-490.

t-Product

☞ Two fourth-order tensors $\mathscr{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3 \times 3}$ and $\mathscr{B} \in \mathbb{R}^{n_2 \times l \times n_3 \times 3}$ then $\mathscr{A} * \mathscr{B}$ represent as follows

$$unfold(\mathscr{A}) = \begin{bmatrix} \mathscr{A}_1 \\ \mathscr{A}_2 \\ \mathscr{A}_3 \end{bmatrix} \quad and \quad Circ(unfold(\mathscr{A})) = \begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_3 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 & \mathscr{A}_3 \\ \mathscr{A}_3 & \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix}$$



Example of product of two tensors

Let $\mathscr{A} \in \mathbb{R}^{3\times3\times2\times2}$ and $\mathscr{B} \in \mathbb{R}^{3\times3\times2\times2}$ Then

$$
\begin{aligned}
\mathscr{A} * \mathscr{B} &= fold\left(\begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix} * \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} \mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2 \\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2 \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12} \\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22} \\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12} \\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22} \\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)
\end{aligned}
$$

☞ The tensor-product of $\mathscr{A}$ and $\mathscr{B}$ eventually reduces to 3×3 matrix multiplication.

Example of product of two tensors

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Let $\mathscr{A} \in \mathbb{R}^{3\times 3\times 2\times 2}$ and $\mathscr{B} \in \mathbb{R}^{3\times 3\times 2\times 2}$ Then

$$
\begin{aligned}
\mathscr{A} * \mathscr{B} &= fold\left(\begin{bmatrix}\mathscr{A}_1 & \mathscr{A}_2\\ \mathscr{A}_2 & \mathscr{A}_1\end{bmatrix} * \begin{bmatrix}\mathscr{B}_1\\ \mathscr{B}_2\end{bmatrix}\right) = fold\left(\begin{bmatrix}\mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2\\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2\end{bmatrix}\right)\\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix}\mathscr{A}_{11} & \mathscr{A}_{12}\\ \mathscr{A}_{12} & \mathscr{A}_{11}\end{bmatrix} * \begin{bmatrix}\mathscr{B}_{11}\\ \mathscr{B}_{12}\end{bmatrix}\right) + fold\left(\begin{bmatrix}\mathscr{A}_{21} & \mathscr{A}_{22}\\ \mathscr{A}_{22} & \mathscr{A}_{21}\end{bmatrix} * \begin{bmatrix}\mathscr{B}_{21}\\ \mathscr{B}_{22}\end{bmatrix}\right)\\ fold\left(\begin{bmatrix}\mathscr{A}_{21} & \mathscr{A}_{22}\\ \mathscr{A}_{22} & \mathscr{A}_{21}\end{bmatrix} * \begin{bmatrix}\mathscr{B}_{11}\\ \mathscr{B}_{12}\end{bmatrix}\right) + fold\left(\begin{bmatrix}\mathscr{A}_{11} & \mathscr{A}_{12}\\ \mathscr{A}_{12} & \mathscr{A}_{11}\end{bmatrix} * \begin{bmatrix}\mathscr{B}_{21}\\ \mathscr{B}_{22}\end{bmatrix}\right)\end{bmatrix}\right)\\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix}\mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12}\\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12}\end{bmatrix}\right) + fold\left(\begin{bmatrix}\mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22}\\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22}\end{bmatrix}\right)\\ fold\left(\begin{bmatrix}\mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12}\\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12}\end{bmatrix}\right) + fold\left(\begin{bmatrix}\mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22}\\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22}\end{bmatrix}\right)\end{bmatrix}\right)
\end{aligned}
$$

☞ The tensor-product of $\mathscr{A}$ and $\mathscr{B}$ eventually reduces to 3×3 matrix multiplication.

Example of product of two tensors

Let $\mathscr{A} \in \mathbb{R}^{3\times3\times2\times2}$ and $\mathscr{B} \in \mathbb{R}^{3\times3\times2\times2}$ Then

$$\mathscr{A} * \mathscr{B} = fold\left(\begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix} * \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} \mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2 \\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2 \end{bmatrix}\right)$$

$$= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)$$

$$= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12} \\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22} \\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12} \\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22} \\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)$$

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Example of product of two tensors

Let $\mathscr{A} \in \mathbb{R}^{3\times3\times2\times2}$ and $\mathscr{B} \in \mathbb{R}^{3\times3\times2\times2}$ Then

$$
\begin{aligned}
\mathscr{A} * \mathscr{B} &= fold\left(\begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix} * \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} \mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2 \\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2 \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12} \\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22} \\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12} \\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22} \\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)
\end{aligned}
$$

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Example of product of two tensors

Let $\mathscr{A} \in \mathbb{R}^{3\times 3\times 2\times 2}$ and $\mathscr{B} \in \mathbb{R}^{3\times 3\times 2\times 2}$ Then

$$
\begin{aligned}
\mathscr{A} * \mathscr{B} &= fold\left(\begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix} * \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} \mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2 \\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2 \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12} \\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22} \\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12} \\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22} \\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)
\end{aligned}
$$

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Example of product of two tensors

Let $\mathscr{A} \in \mathbb{R}^{3\times3\times2\times2}$ and $\mathscr{B} \in \mathbb{R}^{3\times3\times2\times2}$ Then

$$\mathscr{A} * \mathscr{B} = fold\left(\begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix} * \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} \mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2 \\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2 \end{bmatrix}\right)$$

$$= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)$$

$$= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12} \\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22} \\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12} \\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22} \\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)$$

☞ The tensor-product of $\mathscr{A}$ and $\mathscr{B}$ eventually reduces to 3×3 matrix multiplication.

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Example of product of two tensors

Let $\mathscr{A} \in \mathbb{R}^{3\times3\times2\times2}$ and $\mathscr{B} \in \mathbb{R}^{3\times3\times2\times2}$ Then

$$\mathscr{A} * \mathscr{B} = fold\left(\begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix} * \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} \mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2 \\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2 \end{bmatrix}\right)$$

$$= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)$$

$$= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12} \\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22} \\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12} \\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22} \\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)$$

☞ The tensor-product of $\mathscr{A}$ and $\mathscr{B}$ eventually reduces to 3×3 matrix multiplication.

Example of product of two tensors

Let $\mathscr{A} \in \mathbb{R}^{3\times3\times2\times2}$ and $\mathscr{B} \in \mathbb{R}^{3\times3\times2\times2}$ Then

$$
\begin{aligned}
\mathscr{A} * \mathscr{B} &= fold\left(\begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix} * \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} \mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2 \\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2 \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12} \\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22} \\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12} \\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22} \\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)
\end{aligned}
$$

☞ The tensor-product of $\mathscr{A}$ and $\mathscr{B}$ eventually reduces to 3×3 matrix multiplication.

Example of product of two tensors

Let $\mathscr{A} \in \mathbb{R}^{3\times 3\times 2\times 2}$ and $\mathscr{B} \in \mathbb{R}^{3\times 3\times 2\times 2}$ Then

$$
\begin{aligned}
\mathscr{A} * \mathscr{B} &= fold\left(\begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix} * \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} \mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2 \\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2 \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12} \\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22} \\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12} \\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22} \\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)
\end{aligned}
$$

☞ The tensor-product of $\mathscr{A}$ and $\mathscr{B}$ eventually reduces to 3×3 matrix multiplication.

Example of product of two tensors

Let $\mathscr{A} \in \mathbb{R}^{3\times3\times2\times2}$ and $\mathscr{B} \in \mathbb{R}^{3\times3\times2\times2}$ Then

$$
\begin{aligned}
\mathscr{A} * \mathscr{B} &= fold\left(\begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix} * \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} \mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2 \\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2 \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12} \\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22} \\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12} \\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22} \\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)
\end{aligned}
$$

☞ The tensor-product of $\mathscr{A}$ and $\mathscr{B}$ eventually reduces to 3×3 matrix multiplication.

Example of product of two tensors

Let $\mathscr{A} \in \mathbb{R}^{3\times3\times2\times2}$ and $\mathscr{B} \in \mathbb{R}^{3\times3\times2\times2}$ Then

$$\begin{aligned}
\mathscr{A} * \mathscr{B} &= fold\left(\begin{bmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_2 & \mathscr{A}_1 \end{bmatrix} * \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} \mathscr{A}_1 * \mathscr{B}_1 + \mathscr{A}_2 * \mathscr{B}_2 \\ \mathscr{A}_2 * \mathscr{B}_1 + \mathscr{A}_1 * \mathscr{B}_2 \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{22} & \mathscr{A}_{21} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{12} & \mathscr{A}_{11} \end{bmatrix} * \begin{bmatrix} \mathscr{B}_{21} \\ \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right) \\
&= fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{11} + \mathscr{A}_{12} + \mathscr{B}_{12} \\ \mathscr{A}_{12} * \mathscr{B}_{11} + \mathscr{A}_{11} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{21} + \mathscr{A}_{22} * \mathscr{B}_{22} \\ \mathscr{A}_{22} * \mathscr{B}_{21} + \mathscr{A}_{21} * \mathscr{B}_{22} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21} * \mathscr{B}_{11} + \mathscr{A}_{22} * \mathscr{B}_{12} \\ \mathscr{A}_{22} * \mathscr{B}_{11} + \mathscr{A}_{21} * \mathscr{B}_{12} \end{bmatrix}\right) + fold\left(\begin{bmatrix} \mathscr{A}_{11} * \mathscr{B}_{21} + \mathscr{A}_{12} * \mathscr{B}_{22} \\ \mathscr{A}_{12} * \mathscr{B}_{21} + \mathscr{A}_{11} * \mathscr{B}_{22} \end{bmatrix}\right) \end{bmatrix}\right)
\end{aligned}$$

☞ The tensor-product of $\mathscr{A}$ and $\mathscr{B}$ eventually reduces to 3×3 matrix multiplication.

Computations using Fourier Domain

☞ **Example:** Consider two tensors $\mathscr{A} \in \mathbb{R}^{p\times q\times 3}$ and $\mathscr{B} \in \mathbb{R}^{q\times m\times 3}$ then

$$\mathscr{A} * \mathscr{B} = \text{fold}\left(\begin{bmatrix} A_1 & A_3 & A_2 \\ A_2 & A_1 & A_3 \\ A_3 & A_2 & A_1 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}\right) \in \mathbb{R}^{p\times m\times 3}$$

➢ Block circulant matrices can be diagonalized with the help of a normalized DFT matrix.

$$(F_n \otimes I_n)\texttt{bcirc}(\mathscr{A})(F_n^* \otimes I_n) = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{pmatrix} = \text{fft}(\mathscr{A},[\,],3),$$

$$F_n = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{(n-1)} & w^{2(n-1)} & \cdots & w^{(n-1)(n-1)} \end{bmatrix} \in \mathbb{C}^{n\times n},\ w = \exp\left(-\frac{2i\pi}{n}\right)$$

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Computations using Fourier Domain

Let $\mathscr{A} \in \mathbb{R}^{n\times n\times n_3\times\cdots n_p}$, we can write

$$(F_{n_p}\otimes F_{n_{p-1}}\otimes\ldots F_{n_3}\otimes I_n)\cdot\texttt{bcirc}(\mathscr{A})\cdot(F^*_{n_p}\otimes F^*_{n_{p-1}}\otimes\ldots F^*_{n_3}\otimes I_n) = \begin{pmatrix}\Sigma_1 & & & \\ & \Sigma_2 & & \\ & & \ddots & \\ & & & \Sigma_\rho\end{pmatrix}$$

- Which is a block diagonal matrix with ρ blocks, where $\rho = n_3 n_4\cdots n_p$.
- Computation of tensors via the Fourier domain is obtained by systematically reorganizing the tensor into a matrix.
- The matrix computation results will be utilized for tensor computation.

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Algorithm for t-Product

Algorithm 5: Product of two tensors with t-Product

Input $\mathscr{A} \in \mathbb{R}^{m \times n \times p}$, $\mathscr{B} \in \mathbb{R}^{n \times k \times p}$
Compute $\overline{\mathscr{A}} = \text{fft}(\mathscr{A}, [\,], 3);$
Compute $\overline{\mathscr{B}} = \text{fft}(\mathscr{B}, [\,], 3);$
    for $i \leftarrow 1$ to p **do**
        $\overline{\mathscr{C}}(:,:,i) = \overline{\mathscr{A}}(:,:,i)\overline{\mathscr{B}}(:,:,i)$
    end for
Compute $\mathscr{C} = \text{ifft}(\overline{\mathscr{C}}, [\,], 3);$
return $\mathscr{C}$

Identity tensor

Identity tensor

The $n \times n \times n_3 \times \cdots \times n_p$ order-$p(p \geq 3)$ identity tensor I is the tensor such that I_1 is the $n \times n \times n_3 \times \cdots \times n_{p-1}$ order-$(p-1)$ identity tensor and $I_j, j = 2, 3, \cdots n_p$ is the $n \times n \times n_3 \times \cdots \times n_{p-1}$ order-$(p-1)$ zero tensor.

- The $4 \times 4 \times 3 \times 2$ identity tensor $\mathscr{I}$ has the following form

$$\mathscr{I} = fold\left(\begin{bmatrix} \mathscr{I}_1 \\ \mathscr{O}_2 \end{bmatrix}\right) = fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{I}_{11} \\ \mathscr{O}_{12} \\ \mathscr{O}_{13} \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{O}_{21} \\ \mathscr{O}_{22} \\ \mathscr{O}_{23} \end{bmatrix}\right) \end{bmatrix}\right)$$

- Here $\mathscr{I}_{11}$ is the 4×4 identity matrix and $\mathscr{O}_{ij}$ is a tensor all of whose components are zero.

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Transpose of a tensor

If $\mathscr{A} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p}$, then the transpose of $\mathscr{A}$, which is denoted by $\mathscr{A}^T$, is the $n_2 \times n_1 \times n_3 \times \cdots \times n_p$ tensor obtained by tensor transposing each $\mathscr{A}_i$, for $i = 1, 2, \cdots, n_p$ and then reversing the order of the $\mathscr{A}_i$ from 2 through n_p,

$$\mathscr{A}^T = fold\left(\begin{bmatrix} \mathscr{A}_1^T \\ \mathscr{A}_{n_p}^T \\ \mathscr{A}_{n_p-1}^T \\ \vdots \\ \mathscr{A}_2^T \end{bmatrix}\right)$$

Example of a transpose

- Let $\mathscr{A}$ be a $4 \times 4 \times 3 \times 3$ tensor. Then

$$\mathscr{A}^T = fold\left(\begin{bmatrix} \mathscr{A}_1^T \\ \mathscr{A}_3^T \\ \mathscr{A}_2^T \end{bmatrix}\right) = fold\left(\begin{bmatrix} fold\left(\begin{bmatrix} \mathscr{A}_{11}^T \\ \mathscr{A}_{13}^T \\ \mathscr{A}_{12}^T \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{31}^T \\ \mathscr{A}_{33}^T \\ \mathscr{A}_{32}^T \end{bmatrix}\right) \\ fold\left(\begin{bmatrix} \mathscr{A}_{21}^T \\ \mathscr{A}_{23}^T \\ \mathscr{A}_{22}^T \end{bmatrix}\right) \end{bmatrix}\right)$$

- The transpose of the order-4 tensor $\mathscr{A}$ eventually reduces to some 4×4 matrix transpose.

f-diagonal tensor

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f-diagonal

Let $\mathscr{A} = (a_{i_1 \cdots i_p}) \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p}$. Then, $\mathscr{A}$ is called an f-diagonal tensor if $a_{i_1 i_2 \cdots i_p} = 0$ when $i_1 \neq i_2$. Further, $\{a_{i_1 i_2 \cdots i_p | i_1 = i_2}\}$ are called the t-diagonal entries of $\mathscr{A}$.

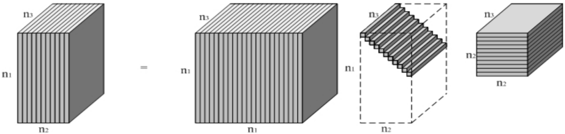
- Here f-diagonal $\longrightarrow$ if each frontal slice is diagonal.
- Let $\mathscr{A}$ be a $3 \times 4 \times 2$ tensor. Then

$$\mathscr{A} = fold\left(\begin{bmatrix} \mathscr{A}_1 \\ \mathscr{A}_2 \end{bmatrix}\right) = fold\left(fold\left(\begin{bmatrix} \mathscr{A}_{11} & 0 & 0 & 0 \\ 0 & \mathscr{A}_{22} & 0 & 0 \\ 0 & 0 & \mathscr{A}_{33} & 0 \\ \mathscr{A}_{11} & 0 & 0 & 0 \\ 0 & \mathscr{A}_{22} & 0 & 0 \\ 0 & 0 & \mathscr{A}_{33} & 0 \end{bmatrix}\right)\right)$$

Tensor Decomposition

Theorem (Tensor Singular Value Decomposition)

Let $\mathscr{A}$ be an $n_1 \times n_2 \times n_3$ real-valued tensor. Then $\mathscr{A}$ can be factored as $\mathscr{A} = \mathscr{U} * \mathscr{D} * \mathscr{V}^T$ where $\mathscr{U}, \mathscr{V}$ are orthogonal $n_1 \times n_1 \times n_3$ and $n_2 \times n_2 \times n_3$, respectively, and $\mathscr{D}$ is a $n_1 \times n_2 \times n_3$ f-diagonal tensor (each frontal slice is diagonal).



Suppose $\mathscr{A}$ is an $n_1 \times n_2 \times n_3$ and F_{n_3} is the $n_3 \times n_3$ DFT matrix. Then $Tcirc(\mathscr{A})$ block circulant matrix, which can be block diagonalized, i.e., there exist matrices $\Sigma_1, \cdots, \Sigma_n$ whose size is $n_1 \times n_2$, possibly with complex entries.

$$F_{n_3} = \frac{1}{\sqrt{n_3}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{n_3-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w^{(n_3-1)} & \cdots & w^{(n_3-1)(n_3-1)} \end{bmatrix} \in \mathbb{C}^{n_3 \times n_3}$$

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- $(F_{n_3} \otimes I_{n_1}) \cdot Tcirc(\mathscr{A}) \cdot (F_{n_3}^* \otimes I_{n_2}) = diag(\Sigma_1, \Sigma_2, \cdots, \Sigma_{n_3})$
- Next, we compute the SVD of each Σ_i. Let $\Sigma_i = U_i \cdot D_i \cdot V_i^T$, for $i = 1, \cdots n_3$

$$\begin{pmatrix} \Sigma_1 & & \\ & \ddots & \\ & & \Sigma_{n_3} \end{pmatrix} = \begin{pmatrix} U_1 & & \\ & \ddots & \\ & & U_{n_3} \end{pmatrix} \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_{n_3} \end{pmatrix} \begin{pmatrix} V_1^T & & \\ & \ddots & \\ & & V_{n_3}^T \end{pmatrix}$$

- Apply $(F_{n_3}^* \otimes I_{n_1})$ to the left and $(F_{n_3} \otimes I_{n_2})$ to the right to the right of each of the block diagonal matrices.
- We now have three block circulant matrices whose first block columns are unfold$(\mathscr{U})$,unfold$(\mathscr{D})$, and unfold$(\mathscr{V}^T)$.

Theorem-1

The Moore-Penrose inverse of a tensor $\mathscr{A} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p}$ exists and is unique.

Ref R. Behera, J. Sahoo, R. Mohapatra, M. Nashid. Computation of Generalized Inverses of Tensors via t-Product. Numer Linear Algebra Appl. 2021 DOI: `10.1002/nla.2416`.

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Color image deblurring

➢ Consider $\mathscr{B} \in \mathbb{R}^{n\times n\times 3}$ is a blurred image, often corrupt with noise.

AIM: To establish a blurred free image.

➢ We write the image blurring problem

$$\mathscr{A} * \mathscr{X} = \mathscr{B},$$

where $\mathscr{A} \in \mathbb{R}^{n\times n\times 3}$ is a blurring tensor.

➢ Need to construct the blurring tensor.

➢ Need to compute $\mathscr{X} = \mathscr{A}^{\dagger} * \mathscr{B}$

Figure: Blurred image

Figure: Reconstruction image

Ref R. Behera, J. Sahoo, R. Mohapatra, M. Nashid. Computation of Generalized Inverses of Tensors via t-Product. Numer Linear Algebra Appl. 2021 DOI: `10.1002/nla.2416`.

Color image deblurring

☞ The color image blurring problem $\mathscr{A} * \mathscr{X} = \mathscr{B}$, with size $n \times n \times 3$

$$\left(A^c \otimes A^h \otimes A^v\right) \begin{pmatrix} \mathbf{vec}(X_{(1)}) \\ \mathbf{vec}(X_{(2)}) \\ \mathbf{vec}(X_{(3)}) \end{pmatrix} = \begin{pmatrix} \mathbf{vec}(B_{(1)}) \\ \mathbf{vec}(B_{(2)}) \\ \mathbf{vec}(B_{(3)}) \end{pmatrix}, \tag{10}$$

Here the operation **vec** take an $n \times n$ matrix and return a $n^2 \times 1$ vector by stacking the columns of the matrix from left to right.

- $A^h \in \mathbb{R}^{n \times n}$ is the horizontal within-channel blurring matrices.
- $A^v \in \mathbb{R}^{n \times n}$ is the vertical within-channel blurring matrices.
- A^c is the cross-channel blurring matrix of size 3×3 as follows:

$$A^c = \begin{pmatrix} c_1 & c_3 & c_2 \\ c_2 & c_1 & c_3 \\ c_3 & c_2 & c_1 \end{pmatrix} \text{ with } \sum_{i=1}^{3} c_i = 1 \text{ and } c_i \in \mathbb{R} \text{ for } i = 1, 2, 3.$$

Ref R. Behera, J. Sahoo, R. Mohapatra, M. Nashid. Computation of Generalized Inverses of Tensors via t-Product. Numer Linear Algebra Appl. 2021 DOI: `10.1002/nla.2416`.

Color image deblurring

Following the circulant structure of the cross-channel blurring matrix, we can write

$$\mathscr{A} * \mathscr{X} * \mathscr{C} = \mathscr{B}, \tag{11}$$

- $\mathscr{A}(:,:,k) = c_k A^v$, for $k = 1,2,3$ with $\mathscr{C}(:,:,1) = (A^h)^T$ and $\mathscr{C}(:,:,j) = \mathbf{0}$, for $j = 2,3$

We construct the blurring tensor $\mathscr{A}$ using the following symmetric banded Toeplitz matrix:

$$A^v_{ij} = A^h_{ij} = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(i-j)^2}{2\sigma^2}}, & |i-j| \leq k \\ 0, & \text{otherwise} \end{cases}$$

Figure: (a) True image (left), (b) Blurred noisy image (middle), (c) Reconstruction image (right).

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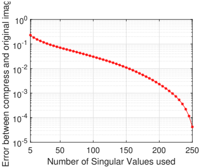
Image compression using SVD

- Since singular values are arranged in decreasing order and thus last terms of singular values have the least effect on the image.
- Consider a positive number k such that $k \leq r = \text{tubal rank} = \#\{i, \mathscr{D}(i,i,:) \neq 0\}$

$$\mathscr{A}_k = \sum_{i=1}^{k} \mathscr{U}(:,i,:) * \mathscr{D}(i,i,:) * \mathscr{V}(:,i,:)^T$$

$$\text{relative error} = \frac{\|\mathscr{A} - \mathscr{A}_{\text{k}}\|_F}{\|\mathscr{A}\|_F}$$

$$\text{where } \|A\|_F^2 = \sum_{i_1=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} |a_{ijk}|^2$$



(a) (b) (c) (d) (e)

Figure: (a) True image; Reconstructions (b) 05 SVs; (c) 15 SVs (d) 25 SVs (e) 50 SVs.

Ref R. Behera, S. Maji, and R. Mohapatra. Comp. Appl. Math, 39, 284, (2020).

Hiding secret color images (video)

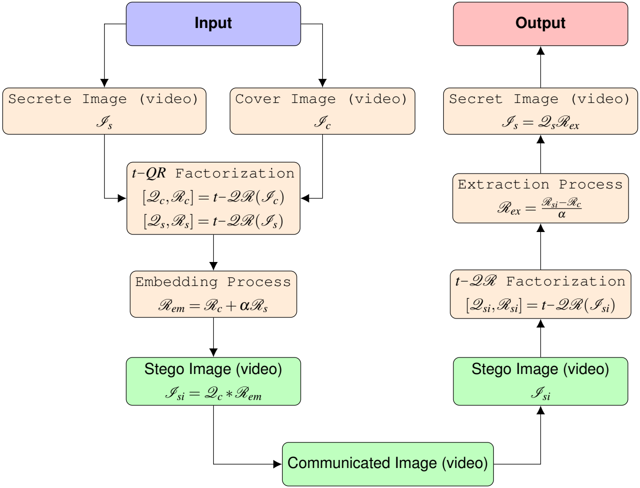
- Steganography is the technique of hiding data within an ordinary file to avoid detection and then extracting it at its destination.
  - $\mathscr{R}$ captures the magnitude/intensity information, the diagonal elements represent the strength or importance of each component
  - $\mathscr{Q}$ captures the directional/orientation information of the image.

(a)

(b)

Figure: (a) Cover image, (b) Input secret image

Tensor-besed steganography for data hiding



Secret color images analysis

Cover image and input secret image

Cover image and input secret image

Stego image and output Secret image

Stego image and output secret image

Secret color video analysis

Input secret video with 20 frames

Cover video with 20 frames

Output secret video with 20 frames

Measuring image quality

- Measuring quality assessment: peak signal-to-noise ratio (PSNR) and structural similarity index (SSIM) technique.
- We define the single PSNR of the color image and color video as:

$$PSNR = \begin{cases} 10 \times \log_{10}\left(\dfrac{max^2}{MSE}\right) & \text{for color image of size } m \times n \times 3. \\ 10 \times \log_{10}\left(\dfrac{max^2}{MSE}\right) & \text{for video of size } m \times n \times 3 \times p. \end{cases}$$

Here *max* represents the highest scale value of the 8-bits grayscale, i.e., $max = 255$.

☞ R. Behera, J. K. Sahoo, and Y. Wei. Numer. Linear Algebra Appl. 32 (1)(2025).

Measuring image quality

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➢ *MSE* represents the mean square error of the tensor slices.

$$MSE = \begin{cases} \frac{1}{3mn}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{3}\left[\mathscr{Z}_{(k)}(i,j) - \mathscr{A}_{(k)}(i,j)\right]^2\right) & \text{for color image.} \\ \frac{1}{3mnp}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{3}\sum_{l=1}^{p}\left[\mathscr{Z}_{(i,l)}(j,k) - \mathscr{A}_{(i,l)}(j,k)\right]^2\right) & \text{for color video.} \end{cases}$$

➢ $\mathscr{Z}_{(k)}(i,j)$ and $\mathscr{A}_{(k)}(i,j)$ are the pixel values of the input and output images at the i,j coordinates of kth frontal slice.

➢ $\mathscr{Z}_{(k,l)}(i,j)$ and $\mathscr{A}_{(k,l)}(i,j)$ are the pixel values of the input color video and output color video at the (i,j) coordinates on k^{th} slice of l^{th} frame.

Measuring image quality

➢ We define structural similarity index measure (SSIM), following three main factors (luminance, contrast, and structure):

$$\mathrm{SSIM}(\mathscr{Z},\mathscr{A}) = l(\mathscr{Z},\mathscr{A})\,c(\mathscr{Z},\mathscr{A})\,s(\mathscr{Z},\mathscr{A}), \tag{12}$$

$$l(\mathscr{Z},\mathscr{A}) = \frac{2\mathscr{F}_{\mathscr{Z}}\mathscr{F}_{\mathscr{A}} + C1}{\mathscr{F}_{\mathscr{Z}}^2 + \mathscr{F}_{\mathscr{A}}^2 + C2}, \quad c(\mathscr{Z},\mathscr{A}) = \frac{2\mathscr{G}_{\mathscr{Z}}\mathscr{G}_{\mathscr{A}} + C2}{\mathscr{G}_{\mathscr{Z}}^2 + \mathscr{G}_{\mathscr{A}}^2 + C2}, \quad s(\mathscr{Z},\mathscr{A}) = \frac{\mathscr{G}_{\mathscr{Z}\mathscr{A}} + C3}{\mathscr{G}_{\mathscr{Z}}\mathscr{G}_{\mathscr{A}} + C3}.$$

$$\mathscr{F}_Z = \begin{cases} \dfrac{\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^3 \mathscr{Z}_{(k)}(i,j)}{3mn} & \text{for color image.} \\ \dfrac{\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^3 \sum_{l=1}^p \mathscr{Z}_{(k,l)}(i,j)}{3mnp} & \text{for color video.} \end{cases}$$

$$\mathscr{G}_Z^2 = \begin{cases} \dfrac{\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^3 \left(\mathscr{Z}_{(k)}(i,j) - \mathscr{F}_{\mathscr{A}}\right)^2}{3mn} & \text{for color image.} \\ \dfrac{\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^3 \sum_{l=1}^p \left(\mathscr{Z}_{(k,l)}(i,j) - \mathscr{F}_{\mathscr{A}}\right)^2}{3mnp} & \text{for color video.} \end{cases}$$

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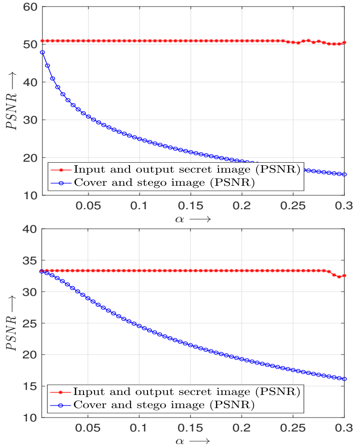
$$\mathscr{G}_{ZA} = \begin{cases} \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{3}\left(\mathscr{Z}_{(k)}(i,j)-\mathscr{F}_{\mathscr{Z}}\right)\left(\mathscr{A}_{(k)}(i,j)-\mathscr{F}_{\mathscr{A}}\right)}{3mn} & \text{for color image} \\ \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{3}\sum_{l=1}^{p}\left(\mathscr{Z}_{(k,l)}(i,j)-\mathscr{F}_{\mathscr{Z}}\right)\left(\mathscr{A}_{(k,l)}(i,j)-\mathscr{F}_{\mathscr{A}}\right)}{3mnp} & \text{for color video.} \end{cases}$$

$$\mathscr{F}_{\mathscr{A}} = \begin{cases} \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{3}\mathscr{A}_{(k)}(i,j)}{3mn} & \text{for color image.} \\ \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{3}\sum_{l=1}^{p}\mathscr{A}_{(k,l)}(i,j)}{3mnp} & \text{for color video.} \end{cases}$$

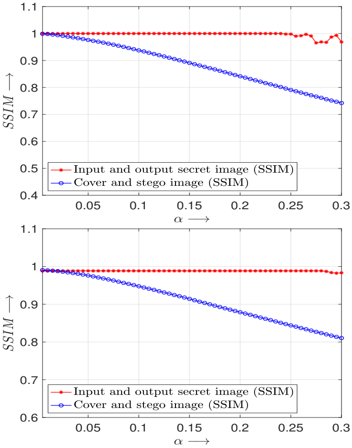
$$\mathscr{G}_{\mathscr{A}}^{2} = \begin{cases} \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{3}\left(\mathscr{A}_{(k)}(i,j)-\mathscr{F}_{\mathscr{A}}\right)^{2}}{3mn} & \text{for color image.} \\ \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{3}\sum_{l=1}^{p}\left(\mathscr{A}_{(k,l)}(i,j)-\mathscr{F}_{\mathscr{A}}\right)^{2}}{3mnp} & \text{for color video.} \end{cases}$$

$$\mathscr{G}_{\mathscr{A}\mathscr{Z}} = \begin{cases} \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{3}\left(\mathscr{A}_{(k)}(i,j)-\mathscr{F}_{\mathscr{A}}\right)\left(\mathscr{Z}_{(k)}(i,j)-\mathscr{F}_{\mathscr{Z}}\right)}{3mn} & \text{for color image.} \\ \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{3}\left(\mathscr{A}_{(k,l)}(i,j)-\mathscr{F}_{\mathscr{A}}\right)\left(\mathscr{Z}_{(k,l)}(i,j)-\mathscr{F}_{\mathscr{Z}}\right)}{3mnp} & \text{for color video.} \end{cases}$$

PSNR and SSIM Results



The PSNR results for different values of α.



The SSIM results for different values of α.

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Thank You!!!